

Relativistic wave equation for one spin-1/2 and one spin-0 particle

D.A. Kulikov*, R.S. Tutik†

Theoretical Physics Department
Dnepropetrovsk National University
13 Naukova str., Dnepropetrovsk 49050, Ukraine

February 1, 2008

Abstract

A new approach to the two-body problem based on the extension of the $SL(2, C)$ group to the $Sp(4, C)$ one is developed. The wave equation with the Lorentz-scalar and Lorentz-vector potential interactions for the system of one spin-1/2 and one spin-0 particle with unequal masses is constructed.

1 Introduction

The relativistic two-body problem has numerous applications in particle and nuclear physics. Because the Bethe-Salpeter equation [1] is exceedingly difficult to solve, even numerically, different approaches to this problem have been developed. They include: reductions of the Bethe-Salpeter equation resulting in the quasipotential approach [2] and the Breit-type equations [3, 4, 5]; relativistic quantum mechanics with constraints [6, 7, 8] that uses a system of two coupled equations describing individual particles; the Barut method [9, 10] for deriving a single two-body wave equation from a field-theoretical action; Lorentz-invariant two-body wave equations having the Schrödinger-like [11] or Dirac-like [12] form.

In the last works the wave functions transform according to the more complicated representations than the one-particle wave functions that can be regarded as involving the extended Lorentz symmetry. The explicit extensions of the Lorentz group, including the symplectic [13] and the general complex [14] ones, have been studied, too.

*kulikov_d_a@yahoo.com

†tutik@ff.dsu.dp.ua

Recently, the extension of the $SL(2, C)$ group to the $Sp(4, C)$ one has allowed us to formulate a new approach to the relativistic two-body problem [15]. The goal of the present work is to apply this technique for constructing the wave equation for the system composed of the spin-1/2 and spin-0 particles with unequal masses.

2 Symplectic space-time extension

The relativistic theory is usually formulated in the Minkowski space with the homogeneous Lorentz group $SO(1, 3)$ as the local symmetry group. However, since the $SO(1, 3)$ group is covered by the $SL(2, C) \equiv Sp(2, C)$ group, the relativistic field theory can equivalently be formulated entirely within the framework of the $Sp(2, C)$ Weyl spinors [16].

Recall that the symplectic $Sp(2l, C)$ group is the group of $2l \times 2l$ matrices with complex elements and determinant equal to one [17]. These matrices act on $2l$ -component Weyl spinors and preserve an antisymmetric bilinear form which plays the role of "metrics" in the spinor space. For the $Sp(4, C)$ group, we denote this form by $\eta_{\alpha\beta} = -\eta_{\beta\alpha}$ ($\alpha, \beta = 1, 2, 3, 4$). Then the $Sp(4, C)$ Weyl spinors φ_α with lower indices and their complex conjugatives $\bar{\varphi}_{\bar{\alpha}} = (\varphi_\alpha)^*$ are related to spinors with upper indices by transformations $\varphi_\alpha = \eta_{\alpha\beta}\varphi^\beta$ and $\bar{\varphi}_{\bar{\alpha}} = \eta_{\bar{\alpha}\bar{\beta}}^*\bar{\varphi}^{\bar{\beta}}$.

Further, there exists one-to-one correspondence between $Sp(2l, C)$ Hermitian spin-tensors of second rank and $(2l)^2$ -component real vectors. In the case of the $Sp(2, C)$ group, they are ordinary Minkowski four-vectors. For the case of $Sp(4, C)$ group, we define the correspondence between the Hermitian spin-tensor, $P_{\alpha\bar{\alpha}}$, and a real vector P_M by

$$P_{\alpha\bar{\alpha}} = \mu_{\alpha\bar{\alpha}}^M P_M, \quad P^M = \frac{1}{4} \tilde{\mu}^{M\bar{\alpha}\alpha} P_{\alpha\bar{\alpha}} \quad (1)$$

where $\mu_{\alpha\bar{\alpha}}^M$ ($M = 1 \div 16$) are matrices of the basis in the space of 4×4 Hermitian matrices and tilde labels the transposed matrix with uppered spinor indices. In what follows, the spinor indices will be omitted when possible.

To clarify the relationship between the discussed vector space and the Minkowski space \mathbf{R}^4 , we represent 16 values of the vector index of P_M through 4×4 combinations of two indices, $M = (a, m)$, with both a and m running from 0 to 3. Then the metrics of the discussed vector

space is reduced to the factorized form

$$g^{MN} \equiv g^{(a,m)(b,n)} = \hat{h}^{ab} h^{mn} \quad (2)$$

where $h^{mn} = \text{diag}(1, -1, -1, -1)$ is the usual Minkowski metrics and $\hat{h}^{ab} = \text{diag}(1, 1, -1, 1)$ is caused by the group extension.

The factorization of the metrics means that the vector from \mathbf{R}^{16} may be decomposed into four Minkowski four-vectors. As a consequence, we can use these 16-component vectors or, equivalently, $Sp(4, C)$ Hermitian spin-tensors to construct the wave equation for a few-body system.

3 Wave equation for a fermion-boson system

Let us consider a system consisted of one spin-1/2 and one spin-0 particle. With the total spin of the system being equal to 1/2, the wave equation must have the form of the Dirac-like equation in which the wave function is represented by a Dirac spinor or, in our case, by two $Sp(4, C)$ Weyl spinors as

$$P_{\alpha\bar{\alpha}}\bar{\chi}^{\bar{\alpha}} = m\varphi_{\alpha}, \quad \tilde{P}^{\bar{\alpha}\alpha}\varphi_{\alpha} = m\bar{\chi}^{\bar{\alpha}} \quad (3)$$

where $P_{\alpha\bar{\alpha}}$ is the $Sp(4, C)$ momentum spin-tensor and m is a mass parameter. According to the splitting of the vector indices, we have

$$P = \mu^{(a,m)} P_{(a,m)} = \Sigma^0 \otimes \sigma^m w_m + \Sigma^1 \otimes \sigma^m p_m + \Sigma^2 \otimes \sigma^m r_m + \Sigma^3 \otimes \sigma^m q_m \quad (4)$$

where w_m, p_m, r_m, q_m are the Minkowski four-momenta and matrices Σ^a, σ^m may be expressed in terms of 2×2 unit matrix I and the Pauli matrices τ^i .

It has been shown [15] that the wave equation (3) describes the fermion-boson system with the equal mass constituents. Now we are going to generalize it to the case of the particles with unequal masses. For this purpose, let us replace the mass parameter in the right hand of Eq.(3) by a suitable matrix term which can be expressed as a combination of direct products of matrices. Though such term breaks the $Sp(4, C)$ symmetry of the wave equation, but the Lorentz $SO(1, 3) \subset Sp(4, C)$ symmetry is retained. It becomes obvious if the second matrix in the direct product is chosen as a unit matrix and the first one is written through the matrices Σ^a , like in Eq.(4). There are two equivalent possibilities, with the matrix Σ^a chosen as $\Sigma^1 = \tau^1$ or $\Sigma^3 = \tau^3$ ($\Sigma^0 = I$ is the trivial choice), that result in the plus sign in the metrics

\hat{h}^{ab} defined by Eq.(2). In view of this we replace the mass parameter as follows

$$m \rightarrow (m_1 + m_2)/2 + \tau^1 \otimes I(m_1 - m_2)/2, \quad (5)$$

so that the additional term vanishes if $m_1 = m_2$.

Thus, the wave equation for the fermion-boson system with unequal masses takes the form

$$P\bar{\chi} = (m_+ + \tau^1 \otimes Im_-)\varphi, \quad \tilde{P}\varphi = (m_+ + \tau^1 \otimes Im_-)\bar{\chi} \quad (6)$$

where $m_{\pm} = (m_1 \pm m_2)/2$.

Now, for elucidating the two-particle interpretation of the proposed equation, we consider the structure of the the $Sp(4, C)$ momentum spin-tensor given by Eq.(4). It should be stressed that the description of the two-particle system requires only two four-momenta whereas the $Sp(4, C)$ momentum spin-tensor corresponds to four four-momenta, collected in a 16-component vector. Therefore the number of the independent components of w_m , p_m , r_m and q_m must be decreased that can be implemented with subsidiary conditions.

In order to derive the subsidiary conditions we transform Eq.(6) to the form of the Klein-Gordon equation. By eliminating $\bar{\chi}$ and using Eq.(4), we obtain

$$(w^2 + p^2 - r^2 + q^2 - \frac{2m_-}{m_+}wp - m_+^2 + m_-^2 + \sum_{A=1}^5 \gamma_A K^A)\varphi = 0 \quad (7)$$

where $w^2 = (w^0)^2 - \mathbf{w}^2$, $p^2 = (p^0)^2 - \mathbf{p}^2$ etc, γ_A are direct products of the Pauli matrices, and K^A are quadratic forms with respect to the four-momenta.

Because in this equation the non-diagonal terms $\gamma_A K^A$ have no analog in the case of the ordinary Klein-Gordon equation, we put $\gamma_A K^A = 0$ that yields

$$\begin{aligned} (m_+^2 - m_-^2)(wp - m_+m_-) - m_+m_-(r^2 - q^2) &= 0, \\ m_+wq - m_-pq &= 0, \\ m_+rp - m_-rw &= 0, \\ rq &= 0, \\ m_+(r^m w^n - r^n w^m - \epsilon^{mnkl} p_k q_l) - m_-(r^m p^n - r^n p^m - \epsilon^{mnkl} w_k q_l) &= 0, \end{aligned} \quad (8)$$

with ϵ^{mnkl} being the totally antisymmetric tensor ($\epsilon^{0123} = +1$).

Thus, the imposed conditions and the Klein-Gordon-like equation set ten components of w_m, p_m, r_m, q_m to be the independent ones. For the connection of these four-momenta with the four-momenta, p_{1m} and p_{2m} , of the constituent particles we assume

$$w_m = \frac{1}{2}(p_{1m} + p_{2m}), \quad p_m = \frac{1}{2}(p_{1m} - p_{2m}), \quad r_m = 0, \quad q_m = 0. \quad (9)$$

Then the only one condition from Eqs.(8) remains non-trivial that reads

$$(wp - m_+m_-) \equiv (p_1^2 - p_2^2 - m_1^2 + m_2^2)/4 = 0. \quad (10)$$

This equality implies that the total spinor wave function does not depend on the relative time of the particles.

Further, the wave equation (6) and the condition (10) can be reduced to the one-particle Dirac and Klein-Gordon equations for the constituents of our system. Indeed, with decomposing the spinor wave functions into the projections

$$\varphi_{\pm} = \frac{1}{2}(1 \pm \tau^1 \otimes I)\varphi, \quad \bar{\chi}_{\pm} = \frac{1}{2}(1 \pm \tau^1 \otimes I)\bar{\chi} \quad (11)$$

which are two-component $Sp(2, C)$ Weyl spinors as well, Eqs.(6) and (10) reduce to two uncoupled sets of equations

$$p_{1m}\sigma^m\bar{\chi}_+ = m_1\varphi_+, \quad p_{1m}\tilde{\sigma}^m\varphi_+ = m_1\bar{\chi}_+ \quad (12)$$

$$(p_2^2 - m_2^2)\varphi_+ = 0, \quad (p_2^2 - m_2^2)\bar{\chi}_+ = 0 \quad (13)$$

and

$$p_{2m}\sigma^m\bar{\chi}_- = m_2\varphi_-, \quad p_{2m}\tilde{\sigma}^m\varphi_- = m_2\bar{\chi}_-, \quad (14)$$

$$(p_1^2 - m_1^2)\varphi_- = 0, \quad (p_1^2 - m_1^2)\bar{\chi}_- = 0, \quad (15)$$

consisted of the free one-particle Dirac equations written in the Weyl spinor formalism [18] and the free Klein-Gordon equations.

Hence it appears that the wave equation (6) supplemented with the subsidiary conditions (8) describes two systems composed of the spin-1/2 and spin-0 particles. These systems differ from each other only in permutation of masses of the particles. As a next step, we must include the potential interaction in our equations.

4 Inclusion of potential interaction

A generally accepted receipt of introducing the interaction consists in the replacement of the four-momenta of the particles in the minimal

manner by the generalized momenta ($p_i^m \rightarrow \pi_i^m = p_i^m - A_i^m$, $i=1,2$), so that each particle is in an external potential of the other. This kind of coupling is referred to as the Lorentz-vector interaction. Another possibility uses the mass-potential substitution, $m_i \rightarrow m_i + S_i$, that corresponds to the Lorentz-scalar interaction.

In our approach the masses and four-momenta of the particles are involved through the quantities w^m, p^m, m_+, m_- . For this reason, we introduce the Lorentz-vector and Lorentz-scalar interactions by the replacements

$$\begin{aligned} w^m &\rightarrow \omega^m = w^m - A^m, & m_+ &\rightarrow M_+ = m_+ + S_+, \\ p^m &\rightarrow \pi^m = p^m - B^m, & m_- &\rightarrow M_- = m_- + S_-. \end{aligned} \quad (16)$$

Here the involved potentials A^m, B^m, S_+, S_- may depend on the coordinates and momenta of the particles but the shape of these potentials is restricted. This restriction is caused by the requirement that the wave equation must be compatible with the subsidiary condition (10) written after the replacements (16) as

$$L \equiv \omega\pi + \pi\omega - M_+M_- - M_-M_+ = 0 \quad (17)$$

A sufficient condition for this compatibility is that the operator L of the subsidiary condition should commute with the operators in the wave equation:

$$[L, \omega_m] \approx 0, \quad [L, \pi_m] \approx 0, \quad [L, M_+] \approx 0, \quad [L, M_-] \approx 0 \quad (18)$$

where the weak equality sign means that the commutator may give an expression proportional to L itself which, on account of Eq.(17), equals to zero.

Because for the quantity wp appearing in Eqs.(16) and (17), we have $[wp, x_m] \neq 0$ but $[wp, x_{\perp m}] = 0$, the conditions in Eqs.(18) require that the potentials depend on the relative coordinate $x_m = x_{1m} - x_{2m}$ only through its transverse with respect to the total momentum part

$$x_{\perp}^m = (h^{mn} - w^m w^n / w^2) x_n \quad (19)$$

where the total momentum w_m is assumed to be a constant of motion.

The simplest solution to the compatibility condition (18) comes from the following ansatz

$$\omega\pi + \pi\omega = 2Cwp, \quad M_+M_- + M_-M_+ = 2Cm_+m_-, \quad (20)$$

where $C = C(x_\perp^2)$ is an arbitrary function. Then the subsidiary condition (17) takes the form of Eq.(10), which describes the case without interaction, that brings at once to vanishing commutators.

Finally, let us derive an explicit form of the wave equation for the fermion-boson system with the potential interactions. With substituting the generalized momenta and the mass-potential terms (16) into Eqs. (4) and (6), we obtain

$$\begin{aligned}(I \otimes \sigma^m \omega_m + \tau^1 \otimes \sigma^m \pi_m) \bar{\chi} &= (M_+ + \tau^1 \otimes IM_-) \varphi, \\ (I \otimes \tilde{\sigma}^m \omega_m + \tau^1 \otimes \tilde{\sigma}^m \pi_m) \varphi &= (M_+ + \tau^1 \otimes IM_-) \bar{\chi}.\end{aligned}\quad (21)$$

Here the quantities $\omega_m, \pi_m, M_+, M_-$ involve the interaction and satisfy the ansatz (20). Using this ansatz, we can introduce both the potential interaction described by the time-component of the Lorentz vector and the confinement potential included in the Lorentz-scalar term or in the spatial part of the Lorentz vector.

Thus, a new approach to the two-body problem based on the extension of the $SL(2, C)$ group to the $Sp(4, C)$ one has been developed. It permits us to construct the relativistic wave equation for the system consisted of spin-1/2 and spin-0 particles with unequal masses, involving the various forms of interaction.

References

- [1] E.E. Salpeter, H.A. Bethe, *Phys. Rev.* **84** (1951) 1232.
- [2] A.A. Logunov, A.N. Tavkhelidze, *Nuovo Cim.* **29** (1963) 380.
- [3] G. Breit, *Phys. Rev.* **34** (1929) 553.
- [4] J. Bijtebier, J. Broekaert, *Nuovo Cim. A* **105** (1992) 625.
- [5] T. Tanaka, A. Suzuki, M. Kimura *Z. Phys. A* **353** (1995) 79.
- [6] H. Sazdjian, *Phys. Rev. D* **33** (1986) 3401.
- [7] H.W. Crater, P. Van Alstine, *Phys. Rev. D* **36** (1987) 3007.
- [8] H.W. Crater, C.-Y. Wong, P. Van Alstine, *arXiv: hep-ph/0603126* (2006).
- [9] A.O. Barut, S. Komy, *Fortschr. Phys.* **33** (1985) 309.
- [10] W. Garczyński, M. Klimek, *Acta Phys. Pol. B* **20** (1989) 755.

- [11] W.I. Fushchych, *Lett. Nuovo Cim.* **10** (1974) 163.
- [12] M. Moshinsky, A.G. Nikitin, *arXiv: hep-ph/0502028* (2005).
- [13] Yu.F. Pirogov, *Phys. Atom. Nucl.* **66** (2003) 136 (*Yad. Fiz.* **66** (2003) 138).
- [14] A.A. Bogush, V.M. Red'kov, *arXiv: hep-th/0607054* (2006).
- [15] D.A. Kulikov, R.S. Tutik, A.P. Yaroshenko, *Phys. Lett. B* **644** (2007) 311, (*arXiv: hep-th/0701271*).
- [16] R. Penrose, W. Rindler, *Spinors and Space-Time: Vol.1* (Cambridge University Press, 1984).
- [17] H. Weyl, *The Classical Groups: Their Invariants and Representations*, (Princeton University Press, 1946).
- [18] E.M. Lifshitz, L.P. Pitaevskii, V.I. Berestetskii, *Quantum Electrodynamics* (Pergamon, Oxford 1982).